Dynamic Derivations for Sequent-Based Logical Argumentation

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Abstract. We introduce a general approach for representing and reasoning with argumentation-based systems. In our framework arguments are represented by Gentzen-style sequents, attacks (conflicts) between arguments are represented by sequent elimination rules, and deductions are made by dynamic proof systems. This framework accommodates different languages and logics in which arguments may be represented, supports a variety of attack relations, and tolerates dynamic changes in the argumentation setting by revising derivations of assertions in light of new information.

Keywords. logical argumentation, sequent calculi, dynamic derivations

1. Introduction

The goal of this paper is to provide a proof theoretical investigation of logical argumentation. Our starting point is that an argument is a pair of a finite set of formulas (Γ, the support set) and a formula (ψ, the conclusion), expressed in an arbitrary propositional language, such that the latter follows, according to some underlying logic, from the former. This abstract approach gives rise to Gerhard Gentzen's well-known notion of a *sequent* [9], extensively used in the context of proof theory. Accordingly, an argument is associated with a sequent of the form $\Gamma \Rightarrow \psi$ and logical argumentation boils down to the exposition of formalized methods for reasoning with these syntactical objects.

In this paper we follow the approach recently taken in [1] that provides a formalized method of constructing arguments by corresponding sequent calculi and expressing attack relations among them in terms of sequent elimination rules. This approach is largely extended in this paper. The primary contribution of the present work is the introduction of a novel *automated machinery* for reasoning with sequent-based arguments. This induces a generic method of drawing conclusions from such arguments, which is tolerant to different logics, languages, and attack rules. For this, we borrow the notion of *dynamic proofs*, used in the context of adaptive logics [2,18], which are intended for explicating actual reasoning in an argumentation framework. Generally, the fact that an argument can be challenged (and possibly withdrawn) by a counter-argument is reflected in dynamic proofs by the ability to consider certain formulas as not derived at a certain stage of the proof, even if they were considered derived in earlier stages of the proof. We believe that this kind of deductive reasoning scheme is particularly useful for logical argumentation, which is non-monotonic and conflict-prone in nature.

2. Sequent-Based Logical Argumentation

Logical argumentation (sometimes called deductive argumentation) is a logic-based approach for formalizing argumentation, disagreements, and entailment relations for drawing conclusions from argumentation-based settings [3,11,14,17].

Definition 1 [7] An *argumentation framework* is a pair $\mathscr{AF} = \langle Args, Attack \rangle$, where *Args* is an enumerable set of elements, called *arguments*, and *Attack* is a relation on $Args \times Args$ whose instances are called *attacks*.

In this paper we follow the sequent-based approach introduced in [1], extending the Besnard-Hunter approach to logical argumentation [3,4]. In particular, unlike [3,4] the underlying language may not be the standard propositional one and the underlying logic may not be classical logic.

In what follows, we shall denote by $\mathscr{L}$ an arbitrary propositional language. Given such a language, we fix a corresponding *logic* (sometimes called the base logic or the core logic), defined as follows.

Definition 2 A (propositional) *logic* for a language $\mathscr{L}$ is a pair $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$, where $\vdash$ is a (Tarskian) consequence relation for $\mathscr{L}$, that is, a binary relation between sets of formulas and formulas in $\mathscr{L}$, satisfying the following conditions:
Reflexivity: if $\psi \in \Gamma$ then $\Gamma \vdash \psi$
Monotonicity: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$
Transitivity: if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \phi$ then $\Gamma, \Gamma' \vdash \phi$

In the sequel, we assume that $\mathscr{L}$ contains the following connectives:

- A unary connective $\neg$ which is a $\vdash$-negation: for every atomic formula p of $\mathscr{L}$ it holds that $p \not\vdash \neg p$ and $\neg p \not\vdash p$,
- A binary connective $\wedge$ which is $\vdash$-conjunctive: for every set of formulas Γ and formulas ψ, ϕ it holds that $\Gamma \vdash \psi \wedge \phi$ iff $\Gamma \vdash \psi$ and $\Gamma \vdash \phi$.

When Γ is finite, we shall denote by $\bigwedge \Gamma$ the conjunction of all the formulas in Γ.

2.1. Arguments As Sequents

In [1] it is argued that what really matters for an argument is that (i) its consequent would logically follow, according to the underlying logic, from the support set, and that (ii) there would be an effective way of constructing and identifying it. This gives rise to the representation of arguments by Gentzen-style sequents, as defined next.

Definition 3 Let $\mathscr{L}$ be a propositional language. An $\mathscr{L}$*-sequent* (a sequent, for short) is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of $\mathscr{L}$- formulas, and $\Rightarrow$ is a symbol that does not appear in $\mathscr{L}$. In what follows we shall denote $\mathsf{Prem}(\Gamma \Rightarrow \Delta) = \Gamma$ and $\mathsf{Cons}(\Gamma \Rightarrow \Delta) = \Delta$.

Proof systems that operate on sequents are called *sequent calculi* [9]. A crucial property shared by all the logics considered in this paper is that they have a sound and complete sequent calculus, that is, a sequent-based proof system $\mathfrak{C}$, such that $\Gamma \vdash \psi$ iff the sequent $\Gamma \Rightarrow \psi$ is provable in $\mathfrak{C}$.

Definition 4 Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic with a corresponding sequent calculus $\mathfrak{C}$, and let Σ be a set of formulas in $\mathcal{L}$. An $\mathfrak{L}$-*argument* based on Σ is an $\mathfrak{L}$-sequent of the form $\Gamma \Rightarrow \psi$, where $\Gamma \subseteq \Sigma$, that is provable in $\mathfrak{C}$. The set of all the $\mathfrak{L}$-arguments that are based on Σ is denoted $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$.

Clearly, we have the following:

Proposition 5 *Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic and let Σ be a set of formulas in $\mathcal{L}$. Then $\Gamma \Rightarrow \psi \in \mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ for some $\Gamma \subseteq \Sigma$ iff $\Sigma \vdash \psi$.*

2.2. Attacks As Elimination Rules

Different attack relations have been considered in the literature for logical argumentation frameworks (see, e.g., [3,10,11]). In our case, attack relations are represented by *sequent elimination rules*, or *attack rules*, which allow to exclude arguments (i.e., sequents) in the presence of counter arguments, as shown in Figure 1. [1]

Defeat:	[Def]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$
Compact Defeat:	[C-Def]	$\dfrac{\Gamma_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$
Direct Defeat:	[D-Def]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \phi \quad \Gamma_2, \phi \Rightarrow \psi_2}{\Gamma_2, \phi \not\Rightarrow \psi_2}$
Compact Direct Defeat:	[CD-Def]	$\dfrac{\Gamma_1 \Rightarrow \neg \phi \quad \Gamma_2, \phi \Rightarrow \psi_2}{\Gamma_2, \phi \not\Rightarrow \psi_2}$
Undercut:	[Ucut]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \neg \bigwedge \Gamma_2 \Rightarrow \psi_1 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}$
Compact Undercut:	[C-Ucut]	$\dfrac{\Gamma_1 \Rightarrow \neg \bigwedge \Gamma_2' \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}$
Direct Undercut:	[D-Ucut]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \gamma_2 \quad \neg \gamma_2 \Rightarrow \psi_1 \quad \Gamma_2, \gamma_2 \Rightarrow \psi_2}{\Gamma_2, \gamma_2 \not\Rightarrow \psi_2}$
Canonical Undercut:	[C-Ucut]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \neg \bigwedge \Gamma_2 \Rightarrow \psi_1 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$
Rebuttal:	[Reb]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \psi_2 \quad \neg \psi_2 \Rightarrow \psi_1 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$
Compact Rebuttal:	[C-Reb]	$\dfrac{\Gamma_1 \Rightarrow \neg \psi_2 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$
Defeating Rebuttal:	[D-Reb]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \psi_2 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$

Figure 1. Sequent elimination rules

[1] Note that the compact version of Direct Undercut is the same as Compact Direct Defeat, the compact version of Canonical Undercut is the same as Compact Defeat, and the compact version of Defeating Rebuttal is the same as Compact Rebuttal.

Note that the conditions of each rule consist of three ingredients: the attacking argument (the leftmost sequent in the condition list of the rule), the attacked argument (the rightmost sequent in the list) and the condition for the attack (expressed by the middle sequent or sequents in the condition list).[2] Conclusions of sequent elimination rules are the discharging of the attacked argument. We denote by $\Gamma \not\Rightarrow \psi$ the elimination (or, the discharging) of the argument $\Gamma \Rightarrow \psi$. Alternatively, $\bar{s}$ denotes the discharging of s.

Definition 6 Let $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ be a set of $\mathfrak{L}$-arguments, $\mathfrak{C}$ a sequent calculus for $\mathfrak{L}$, and $\mathscr{R}$ an elimination rule of the form $\frac{\Gamma_1 \Rightarrow \Delta_1 \ldots \Gamma_n \Rightarrow \Delta_n}{\Gamma_n \not\Rightarrow \Delta_n}$. We say that $s_1 \in \mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ *$\mathscr{R}$-attacks* $s_2 \in \mathsf{Arg}_{\mathfrak{L}}(\Sigma)$, if there is a substitution θ such that $s_1 = \theta(\Gamma_1) \Rightarrow \theta(\Delta_1)$, $s_2 = \theta(\Gamma_n) \Rightarrow \theta(\Delta_n)$, and for each $1 < i < n$, $\theta(\Gamma_i) \Rightarrow \theta(\Delta_i)$ is provable in $\mathfrak{C}$.

The previous definitions yield a Dung-style argumentation framework [7]:

Definition 7 A *logical argumentation framework* for a set of formulas Σ, based on a logic $\mathfrak{L}$ and the rules in $\mathsf{AttackRules}$, is the pair $\mathscr{AF}_{\mathfrak{L}}(\Sigma) = \langle \mathsf{Arg}_{\mathfrak{L}}(\Sigma), \mathit{Attack} \rangle$, where $(s_1, s_2) \in \mathit{Attack}$ iff there is $\mathscr{R} \in \mathsf{AttackRules}$ such that s_1 $\mathscr{R}$-attacks s_2.

In what follows, somewhat abusing the notations, we shall sometimes identify *Attack* with $\mathsf{AttackRules}$.

3. Argumentation by Dynamic Derivations

The uniform representation of rules for constructing arguments and for eliminating them implies that argument derivations may be performed by the same sequent manipulation systems. In our framework this is done by *dynamic proof systems*, which were developed and used in the context of adaptive logics (see [2] and the section on related work below).

Notation 8 A (proof) *tuple* is a quadruple $\langle i, s, \mathsf{J}, \mathsf{C} \rangle$, where i (the tuple's index) is a natural number, s (the tuple's sequent) is either a sequent or an eliminated sequent, J (the tuple's justification) is a string, and C (the tuple's condition) is a set of sequents.

In what follows tuples denote proof steps in dynamic derivations. Tuple's conditions are used to keep track of the assumptions along the derivation. This facilitates the modeling of attacks in the dynamic derivations as defined next.

Definition 9 Let $\mathscr{AF}_{\mathfrak{L}}(\Sigma) = \langle \mathsf{Arg}_{\mathfrak{L}}(\Sigma), \mathit{Attack} \rangle$ be a logical argumentation framework based on a logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$, and let $\mathfrak{C}$ be a sound and complete sequent calculus for $\mathfrak{L}$. A *dynamic $\mathscr{AF}_{\mathfrak{L}}(\Sigma)$-derivation* based on $\mathfrak{C}$ (an $\mathscr{AF}^{\mathfrak{C}}_{\mathfrak{L}}(\Sigma)$-derivation, for short) is a sequence of tuples $\langle i, s, \mathsf{J}, \mathsf{C} \rangle$ (also called *derivation steps* or *proof steps*), where a valid tuple is a tuple of one of the following forms:

- Suppose that $\mathscr{R}$ is an inference rule in $\mathfrak{C}$ of the form

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \ldots \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$$

[2] The only exceptions are the compact rules, in which there are only two conditions representing the attacking sequent and the attacked sequent.

and that θ is an $\mathscr{L}$-substitution such that for every $1 \leq k \leq n$ there is a valid tuple $\langle i_k, s_k, \mathsf{J}_k, \mathsf{C}_k \rangle$ in which s_k is the sequent $\theta(\Gamma_k) \Rightarrow \theta(\Delta_k)$.

Then $\langle l, \theta(\Gamma) \Rightarrow \theta(\Delta), \mathsf{J}, \mathsf{C} \rangle$ is a valid tuple for $l > \max(i_1, \ldots, i_n)$, $\mathsf{J} =$"$\mathscr{R}; i_1, \ldots, i_n$", and $\mathsf{C} = \mathsf{C}_1 \cup \ldots \cup \mathsf{C}_n \cup \{\theta(\Gamma) \Rightarrow \theta(\Delta)\}$.[3]

- Suppose that $\mathscr{R}$ is a sequent elimination rule in *Attack*, which is of the form

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \ldots \ \Gamma_n \Rightarrow \Delta_n}{\Gamma_n \not\Rightarrow \Delta_n}$$

and that θ is an $\mathscr{L}$-substitution such that for every $1 \leq k \leq n$ there is a valid tuple $\langle i_k, s_k, \mathsf{J}_k, \mathsf{C}_k \rangle$ in which s_k is the sequent $\theta(\Gamma_k) \Rightarrow \theta(\Delta_k)$, and $s_1, s_n \in \mathsf{Arg}_{\mathfrak{L}}(\Sigma)$.

Then $\langle l, \theta(\Gamma_n) \not\Rightarrow \theta(\Delta_n), \mathsf{J}, \mathsf{C} \rangle$ is a valid tuple for $l > \max(i_1, \ldots, i_n)$, $\mathsf{J} =$ "$\mathscr{R}; i_1, \ldots, i_n$", and $\mathsf{C} = \{s \in \mathsf{C}_1 \mid \mathsf{Prem}(s) \subseteq \mathsf{Prem}(s_1)\}$.

Note 10 Sequent elimination rules are restricted to attacking and attacked Σ-based arguments only, to prevent situations in which, e.g., $\neg p \Rightarrow \neg p$ attacks $p \Rightarrow p$ although $\Sigma = \{p\}$. This restriction is not imposed on sequent construction rules, whose role is to introduce $\mathfrak{L}$-valid sequents (thus, e.g., when $\mathfrak{L}$ is classical logic, the argument $\Rightarrow r \vee \neg r$ is producible also for $r \notin \mathsf{Atoms}(\Sigma)$. This is due to the fact that $r \Rightarrow r$ is derivable although it is not a Σ-based argument).

Another difference in the applications of these rules is that the condition set of a tuple with a 'positive' (non-eliminated) sequent contains all the condition sequents in the applied rule. In contrast, the condition set of a tuple with an eliminated sequent consists only of the sequents that (i) appear in the condition set of the attacker, and (ii) share premises with the attacker.

As usual in dynamic proof systems, marks are attached to tuples with conditions that are considered unsafe.

Definition 11 Let $\langle T_1, T_2, \ldots, T_n \rangle$ be an $\mathscr{AF}^{\mathfrak{C}}_{\mathfrak{L}}(\Sigma)$-derivation.

- A tuple $T_i = \langle i, s_i, \mathsf{J}_i, \mathsf{C}_i \rangle$ is *marked* (as unsafe) by a tuple $T_j = \langle j, \overline{s_j}, \mathsf{J}_j, \mathsf{C}_j \rangle$, if $j > i$, $s_j \in \mathsf{C}_i$, and $\mathsf{Prem}(s_j) \subseteq \mathsf{Prem}(s_i)$.[4]
- A marked tuple $T_i = \langle i, s_i, \mathsf{J}_i, \mathsf{C}_i \rangle$ is *unmarked* iff for every tuple $T_k = \langle k, \overline{s_k}, \mathsf{J}_k, \mathsf{C}_k \rangle$ by which T_i is marked there is a tuple $T_j = \langle j, \overline{s_j}, \mathsf{J}_j, \mathsf{C}_j \rangle$, such that $k < j$ and $s_j \in \mathsf{C}_k$.

Intuitively, a marking of a tuple is caused by an attack on its sequent or on a sequent in its set of conditions. An unmarking is done when the tuple's argument is defended (that is, when its attackers are counter-attacked). Note that a sequent may be the argument of different tuples with different indices and possibly different conditions and justifications. This means that the same argument may get marked in the context of some tuples but not in the context of other tuples in the same proof. In what follows, when saying that an argument is marked we shall actually refer to its marked tuple, whenever that tuple is clear from the context.

[3] Here, axioms are treated as inference rules without conditions, i.e., they are rules of the form $\frac{}{\Gamma \Rightarrow \Delta}$.
[4] Recall that $\overline{s}$ denotes the discharging of s.

Definition 12 A *marking* $\mathscr{AF}_{\mathfrak{L}}^{\mathfrak{C}}(\Sigma)$*-derivation* is an $\mathscr{AF}_{\mathfrak{L}}^{\mathfrak{C}}(\Sigma)$-derivation of the form $\mathscr{D} = \langle T_1, T_2, \ldots, T_n \rangle$, where:

1. After each production of a valid tuple T_i, marking and unmarking with respect to T_i is done on the sequence $\langle T_1, T_2, \ldots, T_{i-1} \rangle$ according to Definition 11.
2. Tuples of the form $\langle i, \bar{s}, \mathsf{J}, \mathsf{C} \rangle$, obtained by an application of an attack rule $\mathscr{R}$, may be added to the derivation unless there is a tuple $T_k = \langle k, \Rightarrow \neg \bigwedge \Omega, \mathsf{J}_k, \mathsf{C}_k \rangle$ such that $k < i$ and $\Omega \subseteq \bigcup \{\mathsf{Prem}(s') \mid s' \in \mathsf{C}\}$.

The condition in the application of sequent eliminating rules (Item 2 above) is called ISM (Inconsistent Supports Maintenance). It simply blocks the introduction of elimination tuples that are based on premises that have already been shown to be inconsistent.

The outcomes of a marking $\mathscr{AF}_{\mathfrak{L}}^{\mathfrak{C}}(\Sigma)$-derivation are defined next.

Definition 13 A sequent s is *finally derived* in a marking $\mathscr{AF}_{\mathfrak{L}}^{\mathfrak{C}}(\Sigma)$-derivation $\mathscr{D}$ if $T(s) = \langle i, s, \mathsf{J}, \mathsf{C} \rangle$ is an unmarked tuple in $\mathscr{D}$, and $\mathscr{D}$ cannot be extended to a marking $\mathscr{AF}_{\mathfrak{L}}^{\mathfrak{C}}(\Sigma)$-derivation in which $T(s)$ is marked.[5]

Proposition 14 *If a sequent is finally derived in $\mathscr{D}$ then it is finally derived in any extension of $\mathscr{D}$.*

Proof. Suppose that s is finally derived in $\mathscr{D}$ but it is not finally derived in some extension $\mathscr{D}'$ of $\mathscr{D}$. This means that there is some extension $\mathscr{D}''$ of $\mathscr{D}'$ in which s is marked. Since $\mathscr{D}''$ is also an extension of $\mathscr{D}$, we get a contradiction to the final derivability of s in $\mathscr{D}$. $\square$

Definition 15 Let $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ be a logic, $\mathfrak{C}$ a sound and complete sequent calculus for $\mathfrak{L}$, Σ a set of formulas in $\mathscr{L}$, and $\mathscr{AF}_{\mathfrak{L}}(\Sigma) = \langle \mathsf{Arg}_{\mathfrak{L}}(\Sigma), \mathit{Attack} \rangle$ a corresponding argumentation framework. We denote by $\Sigma \mid\!\sim_{\mathfrak{C},\mathit{Attack}} \psi$ that there is a marking $\mathscr{AF}_{\mathfrak{L}}^{\mathfrak{C}}(\Sigma)$-derivation based on $\mathfrak{C}$ and *Attack*, in which $\Gamma \Rightarrow \psi$ is finally derived, for some $\Gamma \subseteq \Sigma$. When $\mathfrak{C}$ and *Attack* are clear from the context we shall sometimes abbreviate $\mid\!\sim_{\mathfrak{C},\mathit{Attack}}$ by $\mid\!\sim$.

4. Examples and Discussion

In this section we use some simple examples for demonstrating dynamic derivations and for presenting entailment relations that are induced by logical argumentation frameworks, as described in the previous section.

4.1. Argumentation Based On Classical Logic

Some logical argumentation frameworks considered in the literature are based on classical logic (see, e.g., [4]). In our case, the corresponding logical argumentation frameworks are of the form $\mathscr{AF}_{\mathrm{CL}}(\Sigma)$, where $\mathrm{CL} = \langle \mathscr{L}, \vdash_{\mathrm{cl}} \rangle$ is classical logic and $\mathscr{L}$ is a propositional language with the standard interpretations for its connectives. The sound and complete sequent calculi used in this case is usually Gentzen's *LK*, given in Figure 2.

[5] A marking $\mathscr{AF}_{\mathfrak{L}}^{\mathfrak{C}}(\Sigma)$-derivation $\mathscr{D}'$ *extends* a marking $\mathscr{AF}_{\mathfrak{L}}^{\mathfrak{C}}(\Sigma)$-derivation $\mathscr{D}$ if, viewed as $\mathscr{AF}_{\mathfrak{L}}^{\mathfrak{C}}(\Sigma)$-derivations (without markings), $\mathscr{D}$ is a prefix of $\mathscr{D}'$.

Axioms: $\psi \Rightarrow \psi$

Structural Rules:

Weakening: $$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Cut: $$\frac{\Gamma_1 \Rightarrow \Delta_1, \psi \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Logical Rules:

$$[\wedge\Rightarrow]\ \frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad [\Rightarrow\wedge]\ \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

$$[\vee\Rightarrow]\ \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta} \qquad [\Rightarrow\vee]\ \frac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}$$

$$[\supset\Rightarrow]\ \frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \supset \varphi \Rightarrow \Delta} \qquad [\Rightarrow\supset]\ \frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \psi \supset \varphi, \Delta}$$

$$[\neg\Rightarrow]\ \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta} \qquad [\Rightarrow\neg]\ \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}$$

Figure 2. The proof system *LK*

Example 16 Consider a logical argumentation for $\Sigma_1 = \{p, \neg p, q\}$, in which Undercut is its single attack rule. Figure 3 shows a marking derivation with respect to this framework. In this figure, asterisks indicate marked tuples and the brackets indicate at which stages the markings are active. To improve readability, we omit the tuple signs in all the derivations in this section. Also, to shorten the derivations a bit, applications of Weakening are sometime omitted, and in the applications of elimination rules we refer only to the attacking and the attacked sequents, omitting the other sequents.[6]

1	$p \Rightarrow p$	Axiom	$\{p \Rightarrow p\}$	$\star$ [3]
2	$\neg p \Rightarrow \neg p$	Axiom	$\{\neg p \Rightarrow \neg p\}$	$\star$ [4]
3	$p \not\Rightarrow p$	Undercut; 2, 1	$\{\neg p \Rightarrow \neg p\}$	
4	$\neg p \not\Rightarrow \neg p$	Undercut; 1, 2	$\{p \Rightarrow p\}$	
...				
i	$\Rightarrow \neg(p \wedge \neg p)$	$(\ldots)$	$\{\ldots\}$	
$i+1$	$q \Rightarrow q$	Axiom	$\{q \Rightarrow q\}$	

Figure 3. A marking derivation for Example 16

[6] Thus, for instance, for applying Undercut in Line 3 of the proof in Figure 3, we implicitly rely on the fact that both $p \Rightarrow \neg\neg p$ and $\neg\neg p \Rightarrow p$ are derivable in *LK*.

The following notes are in order here:

- At Step 3 of the proof, Tuple 1 is marked by Tuple 3. Then, at Step 4 of the proof, Tuple 4 unmarks Tuple 1 and marks Tuple 2 instead. As a consequence, in every extension of the proof above either Tuple 1 or Tuple 2 will be marked. Whenever one extends the proof to unmark one of these tuples, the proof can be further extended to unmark the other tuple. Hence, neither $p \Rightarrow p$ nor $\neg p \Rightarrow \neg p$ is finally derivable, which implies that $\Sigma_1 \not\mid\!\sim p$ and $\Sigma_1 \not\mid\!\sim \neg p$. This is intuitively explained by the fact that the information about p (and about $\neg p$) is not classically consistent, thus it is not reliable, and so it is not derivable.
- Tuple $i+1$, whose sequent is $q \Rightarrow q$, cannot be marked by $p, \neg p \Rightarrow \neg q$ (although the latter is derivable in *LK*), since Tuple i, whose sequent is $\Rightarrow \neg(p \wedge \neg p)$ 'defends' $q \Rightarrow q$. Indeed, in this case Undercut is not applicable due to Condition ISM in Definition 12. Other potential attacks on $q \Rightarrow q$ are blocked by restricting the attacking sequents to Σ-arguments. It follows, then, that $q \Rightarrow q$ is finally derived in the proof above, and so $\Sigma_1 \mid\!\sim q$. This is intuitively explained by the fact that there are no indications that q is related to the inconsistency in Σ_1.

Example 17 Consider the set $\Sigma_2 = \{p, q, \neg(p \wedge q)\}$. This time, none of the formulas in Σ_2 is derivable, since in CL each pair of assertions in Σ_2 attack the third one by Undercut (see Tuples 5, $i+1$, and $j+1$ in the derivation of Figure 4). Again, we denote by asterisks the tuples that are marked at some stage of the derivation (For instance, Tuple 3 is marked between Stage 5 and Stage i. At Stage $i+1$ it is unmarked because one of the conditions of its attacker, Tuple 4, is marked).

1	$p \Rightarrow p$	Axiom	$\{p \Rightarrow p\}$	$\star$
2	$q \Rightarrow q$	Axiom	$\{q \Rightarrow q\}$	$\star$
3	$\neg(p \wedge q) \Rightarrow \neg(p \wedge q)$	Axiom	$\{\neg(p \wedge q) \Rightarrow \neg(p \wedge q)\}$	$\star$
4	$p, q \Rightarrow (p \wedge q)$	$[\Rightarrow \wedge]; 1, 2$	$\{p \Rightarrow p,\ q \Rightarrow q,\ p, q \Rightarrow p \wedge q\}$	$\star$
5	$\neg(p \wedge q) \not\Rightarrow \neg(p \wedge q)$	Undercut; 4, 3	$\{p \Rightarrow p,\ q \Rightarrow q,\ p, q \Rightarrow p \wedge q\}$	
	$\ldots$			
i	$p, \neg(p \wedge q) \Rightarrow \neg q$	$(\ldots); 1, 3$	$\{p \Rightarrow p,\ \neg(p \wedge q) \Rightarrow \neg(p \wedge q), \ldots\}$	$\star$
$i+1$	$q \not\Rightarrow q$	Undercut; i, 2	$\{\ldots\}$	
	$\ldots$			
j	$q, \neg(p \wedge q) \Rightarrow \neg p$	$(\ldots); 2, 3$	$\{q \Rightarrow q,\ \neg(p \wedge q) \Rightarrow \neg(p \wedge q), \ldots\}$	
$j+1$	$p \not\Rightarrow p$	Undercut; j, 1	$\{\ldots\}$	

Figure 4. A marking derivation for Example 17

4.2. Logical Argumentation Based On LP

It is interesting to see how the entailments of Definition 15 behave with respect to base logics other than classical logic. Below, we demonstrate one such case, where the base logic is Priest's 3-valued logic LP [16]. This logic is based on the language of $\{\wedge, \vee, \neg\}$ where $\wedge$, $\vee$, and $\neg$ are, respectively, Kleene's 3-valued conjunction, disjunction and negation whose interpretations are given by the minimum, maximum, and the order reversing functions (respectively) on the lattice $F < I < T$. Intuitively, the truth values T and F correspond to the classical values of truth and falsity. The designated elements in this case (those that represent true assertions) are T and I. A sound and complete proof systems for LP is obtained by adding to *LK* the axiom schemata $\Rightarrow p, \neg p$ and replacing the negation rules $[\neg\Rightarrow]$ and $[\Rightarrow\neg]$ by the logical rules of Figure 5.

$[\neg\neg\Rightarrow]$	$\dfrac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \neg\neg\phi \Rightarrow \Delta}$	$[\Rightarrow\neg\neg]$	$\dfrac{\Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \neg\neg\phi}$
$[\neg\wedge\Rightarrow]$	$\dfrac{\Gamma, \neg\phi \Rightarrow \Delta \quad \Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\phi \wedge \psi) \Rightarrow \Delta}$	$[\Rightarrow\neg\wedge]$	$\dfrac{\Gamma \Rightarrow \Delta, \neg\phi, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\phi \wedge \psi)}$
$[\neg\vee\Rightarrow]$	$\dfrac{\Gamma, \neg\phi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\phi \vee \psi) \Rightarrow \Delta}$	$[\Rightarrow\neg\vee]$	$\dfrac{\Gamma \Rightarrow \Delta, \neg\phi \quad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\phi \vee \psi)}$

Figure 5. Negation rules for LP

Example 18 Let's reconsider the set of assertions $\Sigma_1 = \{p, \neg p, q\}$, this time when LP is the underlying base logic. The marking derivation in Figure 3 is valid in this case as well (for an argumentation framework with Undercut as a sole attack rule), and so, again, $q \Rightarrow q$ is finally derived, while neither $p \Rightarrow p$ nor $\neg p \Rightarrow \neg p$ are finally derivable. Note that this time it is not necessary to include the sequent $\Rightarrow \neg(p \wedge \neg p)$ in the proof, since $p, \neg p \Rightarrow q$ is not derivable in LP, and so $q \Rightarrow q$ is not attackable by any Σ_1-argument.

Example 19 Let us reconsider $\Sigma_2 = \{p, q, \neg(p \wedge q)\}$. This time, whatever attack rules are used, the consequences would be different than those that are obtained when CL is the base logic. Indeed, in LP sequents of the form $p, \neg(p \wedge q) \Rightarrow \neg q$ are *not* derivable, and so proof steps like those described by Tuple i or Tuple j in Figure 4 are not producible. Thus, e.g., in a logical argumentation framework based on LP and Undercut, $p \Rightarrow p$ and $q \Rightarrow q$ are finally derived, while $\neg(p \wedge q) \Rightarrow \neg(p \wedge q)$ is not. The derivation of the first five steps in Figure 4 demonstrates this.

5. Some Properties of $\mid\!\sim$

We now consider some basic properties of entailments of the form $\mid\!\sim$. Below, we fix a logical argumentation framework $\mathscr{AF}_{\mathfrak{L}}(\Sigma) = \langle \mathsf{Arg}_{\mathfrak{L}}(\Sigma), \mathit{Attack} \rangle$ for a set of $\mathscr{L}$-formulas Σ, based on a logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ with a sound and complete sequent calculus $\mathfrak{C}$, and where *Attack* is the set of attacks obtained by the sequent elimination rules in $\mathsf{AttackRules}$.

We start by relating $\mid\!\sim$ and $\vdash$. As the following proposition shows, the latter may be expressed by the former (thus the base consequence relations are obtained by our setting as a particular degenerated case).[7]

[7] Due to short of space proofs for propositions in this section are omitted.

Proposition 20 *If* $\mathsf{AttackRules} = \emptyset$ *then* $\mid\!\sim$ *and* $\vdash$ *coincide.*

Another case where $\mid\!\sim$ and $\vdash$ correlate is the following:

Proposition 21 *If* Σ *is conflict free with respect to* $\mathscr{AF}_{\mathfrak{L}}(\Sigma)$ *(that is, there are no* $s, s' \in \mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ *such that* $(s, s') \in Attack$*) then* $\Sigma \mid\!\sim \psi$ *iff* $\Sigma \vdash \psi$.

In the general case, we have:

Proposition 22 *If* $\Sigma \mid\!\sim \psi$ *then* $\Sigma \vdash \psi$.

Proposition 21 implies, in particular, that $\mid\!\sim$ is *cautiously reflexive*: for every formula ψ such that $\psi \not\vdash \neg\psi$ it holds that $\psi \mid\!\sim \psi$.[8] Yet, as (all) the examples in the previous section show, $\mid\!\sim$ may not be reflexive. These examples also show that in general $\mid\!\sim$ is not monotonic either. For instance, when $\mathfrak{C} = LK$ and $\mathsf{AttackRules}$ consists of any of the attack rules considered in the examples above, we have that $p \mid\!\sim p$ while $p, \neg p \not\mid\!\sim p$.

Proposition 23 *Let* $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ *be a propositional logic and let* $\mid\!\sim$ *be an entailment relation induced by an argumentation framework that is based on* $\mathfrak{L}$*. If* $\vdash$ *is paraconsistent*[9] *then so is* $\mid\!\sim$.

The next proposition shows the "proof invariance" of final derivability.

Proposition 24 *Let* $\mathscr{AF}_{\mathfrak{L}}(\Sigma)$ *be a logical argumentation framework,* $\mathfrak{C}$ *a sequent calculus for* $\mathfrak{L}$*, and* $\mid\!\sim$ *the entailment induced by them. If* $\Sigma \mid\!\sim \varphi$ *then every marking* $\mathscr{AF}^{\mathfrak{C}}_{\mathfrak{L}}(\Sigma)$*-derivation* $\mathscr{D}$ *can be extended to a marking* $\mathscr{AF}^{\mathfrak{C}}_{\mathfrak{L}}(\Sigma)$*-derivation* $\mathscr{D}'$ *such that a tuple with* $\Gamma \Rightarrow \varphi$ *(where* $\Gamma \subseteq \Sigma$*) is finally derived in* $\mathscr{D}'$.

Next, we characterize $\mid\!\sim_{\mathfrak{C},Attack}$-entailments in which *Attack* contains the rules Undercut or Rebuttal. We start with Undercut. When $\mathfrak{C} = LK$, it holds that $\Sigma \mid\!\sim_{LK,\mathrm{Ucut}} \varphi$ iff φ follows in CL from the intersection of the maximal classically consistent subsets of Σ.[10] For the general case we make two assumptions. One is that the base logic satisfies (a variation of) the law of excluded middle (Definition 25). The other assumption is about the consistency preservation of $\mathfrak{C}$: proofs in $\mathfrak{C}$ may not use sequents with inconsistent premises unless this is essential for the proof (Definition 26).

Definition 25 A logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ is called *complete*, if $\vdash \neg(\psi \wedge \neg\psi)$ for every formula ψ in $\mathscr{L}$.

Clearly, both of CL and LP (as well as any other truth functional 3-valued logic with a designated middle element I, for which $\neg I = I$) are complete.

Definition 26 A calculus $\mathfrak{C}$ for $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ is called *normal*, if it contains the axiom $\psi \Rightarrow \psi$ and it is *consistency preserving*: for every set Δ of formulas in $\mathscr{L}$ such that $\not\vdash \neg\bigwedge\Delta'$ for all $\Delta' \subseteq \Delta$, the following holds: If $\Delta \vdash \psi$ then there is a $\mathfrak{C}$-proof of $\Delta \Rightarrow \psi$ where $\{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n\}$ is the set of all used sequents in the proof, and $\not\vdash \neg\bigwedge\Theta$ for all $\Theta \subseteq \Delta \cup \bigcup_{i=1}^{n} \Gamma_i \cup \{\psi\}$.

[8] Note that the condition is indeed required here. For instance, in an argumentation framework based on CL and Undercut it holds that $p \wedge \neg p \not\mid\!\sim p \wedge \neg p$.

[9] That is, for different atoms p, q it holds that $p, \neg p \not\vdash q$.

[10] We shall show this is the full version of the paper.

Proposition 27 *Let $\mathscr{AF}_{\mathfrak{L}}(\Sigma)$ be a logical argumentation framework for a set Σ of formulas in $\mathscr{L}$, based on a complete logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ with a normal, sound and complete calculus $\mathfrak{C}$, and where Undercut is an attack rule. If $\Sigma \mathrel{|\!\sim}_{\mathfrak{C},\mathit{Attack}} \varphi$ then there is a $\mathfrak{C}$-proof of $\Gamma \Rightarrow \varphi$ for some $\Gamma \subseteq \Sigma$, where $\{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n\}$ is the set of all used sequents in the proof (including $\Gamma \Rightarrow \varphi$), and there is no $\Delta \subseteq \Sigma$ for which*

- $\nvdash \neg \bigwedge \Delta'$ *for all* $\Delta' \subseteq \Delta$, *and*
- $\Delta \vdash \neg \bigwedge \Gamma_i'$ *for some* $\Gamma_i' \subseteq \Gamma_i$ $(1 \leq i \leq n)$, *where* $\Gamma_i \subseteq \Sigma$.

The converse of Proposition 27 holds for logics that are not complete and whose sequent calculus is not necessarily normal, but this time we assume that Undercut is the sole attack rule and that Σ is finite.

Proposition 28 *Let $\mathscr{AF}_{\mathfrak{L}}(\Sigma)$ be a logical argumentation framework for a* finite *set Σ of formulas in $\mathscr{L}$, based on a logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ with a sound and complete calculus $\mathfrak{C}$, and where Undercut is the attack rule. It holds that $\Sigma \mathrel{|\!\sim}_{\mathfrak{C},\mathrm{Ucut}} \varphi$ whenever there is a proof in $\mathfrak{C}$ of $\Gamma \Rightarrow \varphi$ for some $\Gamma \subseteq \Sigma$, where $\{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n\}$ is the set of all used sequents in the proof (including $\Gamma \Rightarrow \varphi$), and there is no $\Delta \subseteq \Sigma$ for which*

- $\nvdash \neg \bigwedge \Delta'$ *for all* $\Delta' \subseteq \Delta$, *and*
- $\Delta \vdash \neg \bigwedge \Gamma_i'$ *for some* $\Gamma_i' \subseteq \Gamma_i$ $(1 \leq i \leq n)$, *where* $\Gamma_i \subseteq \Sigma$.

By Proposition 27 and 28 we have a characterization of entailments based on complete logics and Undercut.

Corollary 29 *Let $\mathscr{AF}_{\mathfrak{L}}(\Sigma)$ be a logical argumentation framework for a finite set Σ of formulas in $\mathscr{L}$, based on a complete logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ with a normal, sound and complete calculus $\mathfrak{C}$, and where Undercut is the attack rule. Then $\Sigma \mathrel{|\!\sim}_{\mathfrak{C},\mathrm{Ucut}} \varphi$ if and only if there is a proof in $\mathfrak{C}$ of $\Gamma \Rightarrow \varphi$ for some $\Gamma \subseteq \Sigma$, where $\{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n\}$ is the set of all used sequents in the proof, and there is no $\Delta \subseteq \Sigma$ for which*

- $\nvdash \neg \bigwedge \Delta'$ *for all* $\Delta' \subseteq \Delta$, *and*
- $\Delta \vdash \neg \bigwedge \Gamma_i'$ *for some* $\Gamma_i' \subseteq \Gamma_i$ $(1 \leq i \leq n)$, *where* $\Gamma_i \subseteq \Sigma$.

Let us turn to logical frameworks with Rebuttal. It is possible to show propositions that are dual to 27 and 28, where Undercut is replaced by Rebuttal, and the condition

- There is no $\Delta \subseteq \Sigma$ for which $\Delta \vdash \neg \bigwedge \Gamma_i'$ for some $\Gamma_i' \subseteq \Gamma_i$ $(1 \leq i \leq n)$ where $\Gamma_i \subseteq \Sigma$,

is replaced by the condition

- There is no $\Delta \subseteq \Sigma$ for which $\Delta \vdash \neg \psi_i$ for some $1 \leq i \leq n$ where $\Gamma_i \subseteq \Sigma$ and $\Delta_i = \{\psi_i\}$.

Thus, we have a characterization of entailments based on complete logics and Rebuttal:

Proposition 30 *Let $\mathscr{AF}_{\mathfrak{L}}(\Sigma)$ be a logical argumentation framework for a finite set Σ of formulas in $\mathscr{L}$, based on a complete logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ with a normal, sound and complete calculus $\mathfrak{C}$, and where Rebuttal is the attack rule. Then $\Sigma \mathrel{|\!\sim}_{\mathfrak{C},\mathrm{Reb}} \varphi$ if and only if there is a proof in $\mathfrak{C}$ of $\Gamma \Rightarrow \varphi$ for some subset $\Gamma \subseteq \Sigma$, where $\{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n\}$ is the set of all used sequents in the proof, and there is no subset $\Delta \subseteq \Sigma$ for which*

- $\nvdash \neg \bigwedge \Delta'$ *for all* $\Delta' \subseteq \Delta$, *and*
- $\Delta \vdash \neg \psi_i$ *for some* $1 \leq i \leq n$ *where* $\Gamma_i \subseteq \Sigma$ *and* $\Delta_i = \{\psi_i\}$.

6. Conclusion and Related Work

Many different approaches to logical argumentation have been introduced in the literature. This includes formalisms that are based on classical logic [3,4], defeasible reasoning [11,12,17] abstract argumentation and the ASPIC$^+$ framework [15], assumption-based argumentation [8], default logic [13], situation calculus [5], and so forth.

Due to space limitation a detailed comparison to these and other approaches is delayed to an extended version of this paper. It should be mentioned, however, that like ASPIC$^+$ our approach provides a very flexible environment for logical argumentation, as it leaves open the choices of the underlying language, the core logic, and the adequate calculus. This flexibility carries on to the representation of arguments that avoids the minimality and consistency constraints posed on the premises of arguments in [3].

The connection between dynamic derivations and Dung-style argumentation semantics [7] has to be investigated. We refer to [19] for some results in this direction, concerning deontic logics. Of a special interest are status assignments as in [6,12]. Such assignments may be related to different statuses of sequents in dynamic derivations, such as finally derived tuples, finally defeated tuples, and tuples whose markings are fluctuating.

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